

**Note**

**A Splitting Scheme for the Numerical Solution  
of a One-Dimensional Vlasov Equation**

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A splitting scheme for the numerical integration in phase-space of the one-dimensional Vlasov equation has been recently proposed by Cheng and Knorr [1]. In order to test and to compare the accuracy and stability of this method with respect to other methods (especially with respect to results recently obtained using a hybrid model proposed by Denavit [2]), and with the intent of extending it to the numerical integration of a two-dimensional Vlasov equation, a numerical code has been developed using this new method. To make the scheme more economical, the interpolation procedure presented by Cheng and Knorr [1] has been slightly modified; this optimization is essential if one intends to solve the two-dimensional Vlasov equation.

The pertinent equations in dimensionless form are

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E(x, t) \frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\partial E / \partial x = \int_{-\infty}^{\infty} f \, dv - 1, \tag{2}$$

where the symbols have their conventional meaning.

The method used by Cheng and Knorr [1] consisted in splitting up the free-streaming term and acceleration term in Eq. (1), and solving first the free-streaming term

$$(\partial f / \partial t) + v(\partial f / \partial x) = 0 \tag{3}$$

and then solving the acceleration term

$$(\partial f / \partial t) + E(x, t)(\partial f / \partial v) = 0. \tag{4}$$

The integration of Eqs. (3) and (4) was reduced to the following shifting sequence of the distribution function [1]

$$f^*(x, v) = f^n(x - v \Delta t / 2, v), \tag{5}$$

$$f^{**}(x, v) = f^*(x, v - E(x) \Delta t), \tag{6}$$

$$f^{n+1}(x, v) = f^{**}(x - v \Delta t / 2, v), \tag{7}$$

where  $f^n$  denotes the value of the distribution function at a time  $t = n \Delta t$ . The electric field used in Eq. (6) for the shift was calculated from  $f^*(x, v)$  given in Eq. (5). The results obtained by this method are of second order in  $\Delta t$ .

The successive shifts in Eqs. (5)–(7) were effected by interpolating the value of  $f$  in the  $x$  and  $v$  directions. A Fourier interpolation technique was used for the  $x$  direction, and a cubic spline interpolation technique was used for the  $v$  direction [1]. Interpolating  $N$  values of  $f$  with the Fourier interpolation technique requires a computational effort proportional to  $N^2$ . The results reported in [1] used  $N = 8,16$  and consequently the computation time remained reasonable. However, when  $N$  becomes large (we take  $N = 64$  in our present calculations), or when dealing with problems of higher dimensionality it is more advantageous to use periodic splines [4], where the number of operations grows with  $N$  only. For this reason, in the present code a cubic spline interpolation has been used to calculate the successive shifts in Eqs. (5)–(7).

As a test case we consider the linear Landau damping and the subsequent amplitude oscillations which develops for small values of  $\gamma/\omega_b$  [5] ( $\gamma$  is the linear Landau damping rate and  $\omega_b$  is the bounce frequency of the trapped particles). The initial condition is

$$f(x, v, 0) = f_0(v)(1 + \alpha \cos kx) \quad (8)$$

where  $\alpha = 0.01$ , the periodic length of the system is  $L = 20\pi$ ,  $k = 2\pi n/L$  with  $n = 3$ ,  $\Delta t = \frac{1}{8}$ , and  $f_0(v) = (2\pi)^{-1/2} \exp(-v^2/2)$  with  $V_{\max} = 5.0$ . We are using a number of points  $N = 64$  in the  $x$  direction, and a number of points  $2M = 128$  in the  $v$  direction; this is the equivalent of using 8192 “particles.” The computation results are shown in Figs. (1)–(3). In Fig. 1, the electric energy is plotted, against time, on a linear scale; it shows the initial exponential decay due to Landau damping, followed by the amplitude oscillation predicted by the theory [5]. The recurrence effect, occurring at  $T_R = 2\pi/(k \Delta v) \approx 266$ , is clearly apparent on Figs. 1 and 2

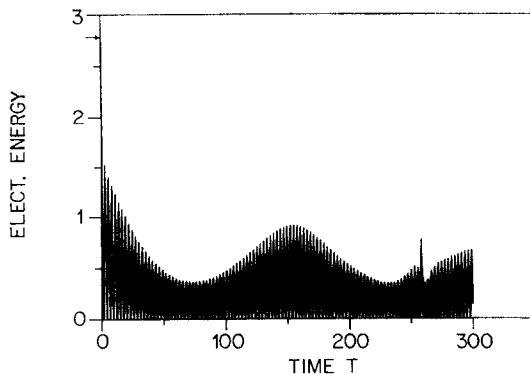


FIG. 1. Plot of the electric energy (in arbitrary units) against time (in units  $\omega_b^{-1}$ ). The arrow indicates the electric energy at  $t = 0$ .

(a similar recurrence time can be obtained by no less than 1600 Hermite polynomials when using the Hermite polynomials expansion method). Figure 2 shows the electric field excited at  $t = 0$  with  $n = 3$  plotted on a logarithmic scale; it follows closely the oscillating behavior of Fig. 1. The calculated initial numerical values for  $\omega/\omega_p$  and  $\gamma/\omega_p$  are, respectively, 1.165 and 0.0125 (while for  $k = 0.3$ , the corresponding theoretical values are 1.1598 and 0.01262). The bounce frequency  $\omega_b$  of the trapped particles is  $\omega_b/\omega_p = (\alpha/2)^{1/2} = 0.0707$ , hence  $\gamma/\omega_b = 0.176$ , which is well within the range of values where amplitude oscillations should occur [5]. The relative error in energy conservation at  $\omega_p t = 250$ , just before recurrence occurs (i.e., after 2000 time-step) was about  $10^{-3}$  (we are using single precision arithmetic); the computational

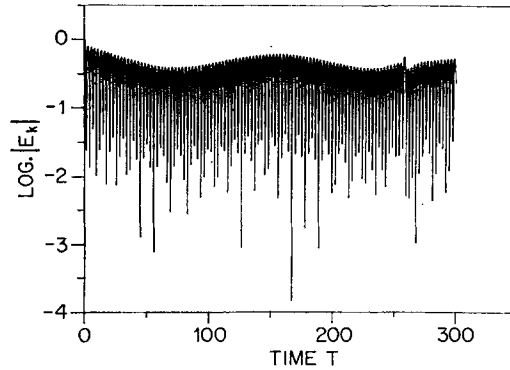


FIG. 2. Plot on a logarithmic scale of the absolute value of the Fourier mode  $E_k$  with  $k = 2\pi n/L$  ( $n = 3$ ,  $L = 20\pi$ ), against time.

time required (CPU time) up to  $\omega_p t = 250$  was approximately 4100 seconds which corresponds to a computational effort of 0.25 msec per particle per time-step ( $\Delta t = \frac{1}{8}$ ), using an IBM 370/155. It should be noted that the results recently reported by Shoucri and Gagné [6], with the Hermite polynomials expansion method, used a time-step  $\Delta t = 1/16$  in order to avoid numerical instabilities which developed when using larger values of  $\Delta t$ .

Results have been recently reported [3], where the numerical calculation has been carried out up to  $\omega_p t \approx 280$ . The calculations used a hybrid model proposed by Denavit [2] which can reduce the fluctuations inherent to the particle simulation model. The calculations reported in [3] used 8192 particles, which makes them equivalent to our present work; in these calculations, however, a smoothing operation had to be repeated every 16 time-steps in order to combat a streaming instability inherent to the numerical code. Contrasting with this, there is no smoothing operation at all applied in our present calculations; if a smoothing of the ripples of the distribution function had been applied as was effected in [1], our present results would have been obtained using much less than the equivalent of 8192 particles. Most important in our results is the absence of problems associated with thermal fluctuations

of the macroscopic quantities; this makes the study of the behavior of very small amplitude waves possible in a very accurate manner.

The computed distribution function presented in Fig. 3 shows the existence of a perturbation in the region around  $v = \omega/k \simeq 3.86$ ; a bump is formed which reaches a maximum height for  $\omega_p t \simeq 73$  and  $\omega_p t \simeq 230$ , two values which, in Fig. 1, correspond respectively to the first and the second amplitude minima. The formation of the bump is accompanied by a positive slope in the distribution function which causes the growth of waves with phase velocities lying in that region of the bump, and, also, causes the amplitude in Figs. 1-2 to reach a maximum and then decrease when the

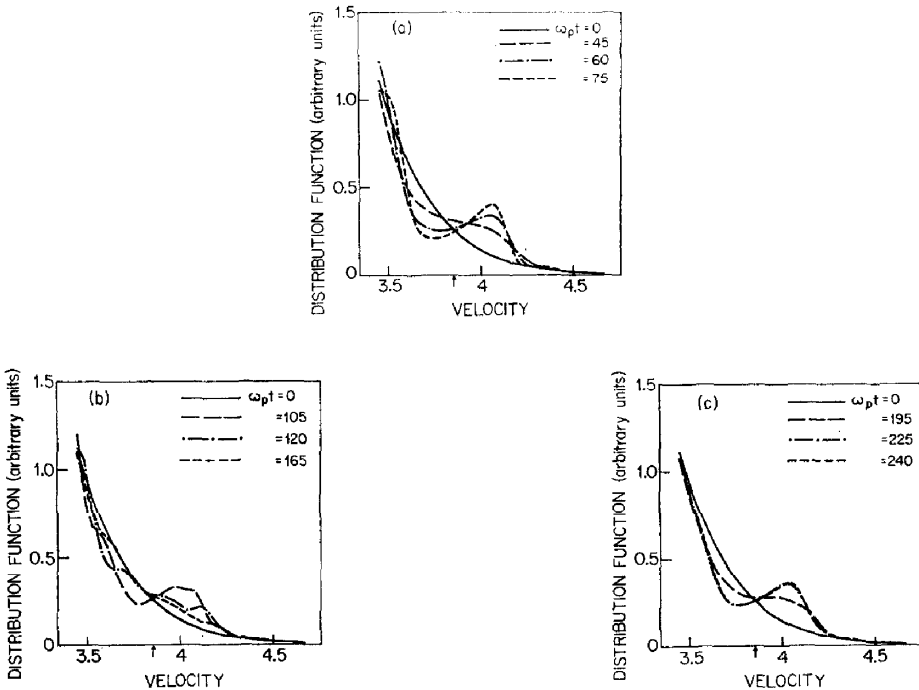


FIG. 3. Temporal behavior of the spatially-averaged distribution function. The phase velocity of the wave excited at  $t = 0$  is indicated by an arrow.

slope of the distribution function turns to negative. The height of the bump, for increasing values of  $\omega_p t$ , becomes progressively smaller on its reappearance because of phase mixing of the trapped particles. All the excited modes remained smaller than the mode  $n = 3$  by at least two orders of magnitude, so that most of the electric energy was concentrated in the  $n = 3$  mode. The curves in Figs. 1-3 show that the oscillations are occurring around an asymptotic value close to the plateau given by the curve for  $\omega_p t = 195$  in Fig. 3.

The displacement of the averaged distribution function, from its equilibrium

Maxwellian value, is very small (the peak in Fig. 3 at  $\omega_p t = 240$  is only about  $0.2 \times 10^{-3}$  higher than the Maxwellian curve represented by the full line).

The results presented indicate that the code is stable, accurate and efficient. The code is actually being used to study the behavior of a monochromatic wave for much longer times [7] than has been done here. Work is also in progress to use it in a two-dimensional scheme for the numerical solution of Vlasov equation [8], and to compare the results thus obtained, with those recently reported using the Hermite polynomials expansion method [9]. We intended in the present note to illustrate the method using an amount of information equivalent to the one used in [3] in order to make a qualitative comparison possible.

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